Inverse problems for nonlocal and nonlinear wave equations

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Nonlocal nonlinear wave equations

We study inverse problems for nonlinear nonlocal wave equations

$$\partial_t^2 u + (-\Delta)^s u + f(x,u) = 0 \quad \text{in } \Omega_T := \Omega \times (0,T),$$

where

- The fractional Laplacian:

$$(-\Delta)^s : H^t(\mathbb{R}^n) \to H^{t-2s}(\mathbb{R}^n)$$
$$(-\Delta)^s u = \mathcal{F}^{-1}\left(|\xi|^{2s}\widehat{u}\right)$$

- $\Omega \subset \mathbb{R}^n$ is bounded Lipschitz domain,
- $0 < s < 1$,
- exterior Dirichlet data in $(\Omega_e)_T := (\mathbb{R}^n \setminus \overline{\Omega})_T$,
- $f(x,u)$ is a polyhomogeneous nonlinearity.

The inverse problem

Given exterior Dirichlet data $\varphi \in C_c^\infty(W)$, $W \subset \Omega_e$, consider

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + f(x,u) = 0 & \text{in } \Omega_T, \\ u = \varphi & \text{in } (\Omega_e)_T, \\ u(0) = \partial_t u(0) = 0 & \text{in } \Omega. \end{cases}$$

The associated Dirichlet-to-Neumann map is

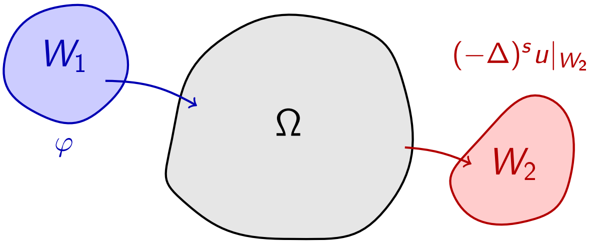
$$\Lambda_f : \varphi \mapsto (-\Delta)^s u\big|_{(\Omega_e)_T}$$

$$\langle \Lambda_f \varphi, \psi \rangle := \int_{\mathbb{R}^n_T} (-\Delta)^{\frac{s}{2}} u \, (-\Delta)^{\frac{s}{2}} \psi dxdt$$

Main question:

Can one uniquely recover the nonlinearity $f(x,\tau)$ from partial exterior measurements?

Measurement setup



$$\Omega_e = \mathbb{R}^n \setminus \Omega$$

Main result

Assume

$$f(x,u) \sim \sum_{k=1}^{\infty} f_k(x,u),$$

where the functions $f_k(x,\cdot)$ are homogeneous of order $r_k + 1$, $0 < r_1 < r_2 < \ldots$.

Theorem (Lin-T-Zimmermann)

Suppose two nonlinearities $f^{(1)}$ and $f^{(2)}$ produce the same partial DN map

$$\Lambda_{f^{(1)}}\varphi\big|_{(W_2)_T} = \Lambda_{f^{(2)}}\varphi\big|_{(W_2)_T}$$

for all small exterior data supported in $(W_1)_T$.
Then all coefficients in the polyhomogeneous expansion agree:

$$f_k^{(1)}(x,s) = f_k^{(2)}(x,s), \quad k = 1, 2, \ldots$$

Background and context

- Ghosh, Salo, Uhlmann (2020): UCP and Runge in L^2 for $(-\Delta)^s u + qu = 0$.
- Rüland, Salo (2020): Stability and Runge approximation in $\widetilde{H}^s$
- Many results for elliptic and parabolic nonlocal equations: Cekic, Covi, Feizmohammadi, Ghosh, Kaltenbacher, Krupchyk, C-L Lin, Y-H Lin, Xiao, Railo, Zimmermann, and many others...
- P-Z Kow, Y-H Lin, and J-N Wan (2022): Nonlocal wave equation $(\partial_t^2 + (-\Delta)^s + q)u = 0$, $q \in L^\infty$

Example: peridynamics and nonlocal models

Nonlocal wave equations appear naturally in:

- peridynamics
- anomalous diffusion-wave phenomena,
- long-range interaction models,
- nonlocal continuum mechanics.

General peridynamic equation of motion[1]:

$$\partial_t^2 u = \int_{\mathbb{R}^n} f(u(y,t) - u(x,t); y - x)dy + b(x,t,u)$$

The integral on the RHS can be, for example,

$$(-\Delta)^s u(x) = c_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\,dy.$$

[1] S.A. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. J.Mech.Phys.Solid., 48(1), 175–209, 2000.

Function spaces and weak solutions

We use the Sobolev spaces (Bessel potential spaces)

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2 : (1 + |\xi|^2)^{s/2} \widehat{u}(\xi) \in L^2 \right\}.$$

For the interior domain:

$$\widetilde{H}^s(\Omega) = \overline{C_c^\infty(\Omega)}^{H^s(\mathbb{R}^n)}$$

with the dual space

$$(\widetilde{H}^s(\Omega))^* = H^{-s}(\Omega).$$

Weak solutions

$$\begin{cases} (\partial_t^2 + (-\Delta)^s)u + qu = F, & \text{in } \Omega_T \\ u = \varphi & \text{in } (\Omega_e)_T \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \text{in } \Omega \end{cases}$$

Theorem (Weak solutions)

If $u_0 \in \widetilde{H}^s(\Omega)$, $u_1 \in L^2(\Omega)$, $q \in L^p(\Omega)$, $F \in L^2(\Omega_T)$, $\varphi \in C_c^\infty(W_1)$, then the nonlocal wave equation has unique weak solution $u \in C([0,T]; H^s(\mathbb{R}^n)) \cap C^1([0,T]; L^2(\mathbb{R}^n))$.

We always take $p \geq 1$ satisfying

$$\begin{cases} p \geq \frac{n}{2s}, & 2s < n, \\ p > 2, & 2s = n, \\ p = 2, & 2s > n. \end{cases}$$

Very weak solutions

$$\begin{cases} (\partial_t^2 + (-\Delta)^s)u + qu = F, & \text{in } \Omega_T \\ u = 0 & \text{in } (\Omega_e)_T \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \text{in } \Omega \end{cases}$$

Theorem (Very weak solutions)

If $u_0 \in \widetilde{L}^2(\Omega)$, $u_1 \in H^{-s}(\Omega)$, $q \in L^p(\Omega)$, $F \in L^2(0, T; H^{-s}(\Omega))$, $\varphi \in C_c^\infty(W_1)$, then the nonlocal wave equation has a unique very weak solution $u \in C([0, T]; \widetilde{L}^2(\Omega)) \cap C^1([0, T]; H^{-s}(\Omega))$

Very weak solutions

$$\begin{cases} \partial_t^2 u + (-\Delta)^s u + qu = F & \text{in } \Omega_T, \\ u = 0 & \text{in } (\Omega_e)_T, \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega, \end{cases}$$

$u_0 \in \widetilde{L}^2(\Omega)$, $u_1 \in H^{-s}(\Omega)$, $q \in L^p(\Omega)$, $F \in L^2(0,T;H^{-s}(\Omega))$. A very weak solution is a function

$$u \in C([0,T];\widetilde{L}^2(\Omega)) \cap C^1([0,T];H^{-s}(\Omega))$$

satisfying

$$\int_0^T \langle u, G\rangle_{L^2}\, dt = \int_0^T \langle F, v\rangle\, dt + \langle u_1, v(0)\rangle - \langle u_0, \partial_t v(0)\rangle_{L^2},$$

for all $G \in L^2(0,T;\widetilde{L}^2(\Omega))$.
Here v is the weak solution of the adjoint equation

$$\begin{cases} \partial_t^2 v + (-\Delta)^s v + qv = G & \text{in } \Omega_T, \\ v = 0 & \text{in } (\Omega_e)_T, \\ v(T) = \partial_t v(T) = 0 & \text{in } \Omega. \end{cases}$$

Linearized problem and Runge approximation

$$\begin{cases} (\partial_t^2 + (-\Delta)^s)u + qu = 0, & \text{in } \Omega_T \\ u = \varphi & \text{in } (\Omega_e)_T \\ u(0) = u_0,\ \partial_t u(0) = u_1 & \text{in } \Omega \end{cases}$$

Define the Runge set

$$\mathcal{R}_W = \{u_\varphi - \varphi : \varphi \in C_c^\infty((W)_T)\}, \quad W \subset (\Omega_e)_T \text{ open},$$

Theorem (Runge approximation, Lin-T-Zimmermann)

The set $\mathcal{R}_W$ is dense in $L^2(0, T; \widetilde{H}^s(\Omega))$.

Idea of the Runge proof

Assume

$$F \in L^2(0, T; H^{-s}(\Omega))$$

is orthogonal to the Runge set:

$$\langle F, v \rangle = 0 \qquad \text{for all } v \in \mathcal{R}_W.$$

Solve the dual problem backward in time:

$$\begin{cases} \partial_t^2 w + (-\Delta)^s w + qw = F, \\ w|_{\Omega_e} = 0, \\ w(T) = \partial_t w(T) = 0. \end{cases}$$

Testing against Runge solutions gives for all $\varphi \in C_c^\infty(W_T)$

$$\int_0^T \langle w, (-\Delta)^s \varphi \rangle_{L^2(\Omega)} \, dt = 0 \quad \Longrightarrow \quad (-\Delta)^s w = 0 \text{ in } W_T.$$

Since also $w = 0$ in $(\Omega_e)_T$

Unique continuation property (Ghosh-Salo-Uhlmann, 2016-20)

Let $0 < s < 1$ and $u \in H^{-r}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$. If $u = (-\Delta)^s u = 0$ in some open set, then $u \equiv 0$.

$$\implies w \equiv 0.$$

Therefore the dual equation

$$\partial_t^2 w + (-\Delta)^s w + qw = F$$

implies

$$F = 0$$

Hence the orthogonal complement of $\mathcal{R}_W$ is trivial, so by Hahn–Banach $\mathcal{R}_W$ is dense in

$$L^2(0, T; \widetilde{H}^s(\Omega)).$$

Nemytskii nonlinearities

Assume $s > 0$ is non-integer let

$$f : \Omega \times \mathbb{R} \to \mathbb{R}$$

be a Carathéodory function satisfying $f(\cdot, 0) \in L^2(\Omega)$ and

$$|\partial_\tau f(x,\tau)| \lesssim a(x) + |\tau|^r, \quad a \in L^p(\Omega)$$
$$\int_0^\tau f(x,\rho)d\rho \geq -C_1, \quad C_1 > 0$$

with

$$\begin{cases} 0 \leq r < \infty, & \text{if } 2s \geq n, \\ 0 \leq r \leq \frac{2s}{n-2s}, & \text{if } 2s < n, \end{cases}$$

Polyhomogeneous expansion

We call f *asymptotically polyhomogeneous*, denoted

$$f(x,\tau) \sim \sum_{k\geq 1} f_k(x,\tau),$$

if each f_k is (r_k+1)-homogeneous in τ and for every $N \geq 2$,

$$f(x,\tau) - \sum_{k=1}^{N-1} f_k(x,\tau) = \mathcal{O}(|\tau|^{r_N+1}) \qquad (|\tau| \leq 1),$$

while

$$f(x,\tau) = \mathcal{O}(|\tau|^{r_\infty+1}) \qquad (|\tau| > 1),$$

with $r_1 < r_2 < \ldots < r_\infty < \infty$.

$$f(x,u) = q_1(x)|u|^{r_1}u + \ldots + q_\infty(x)|u|^{r_\infty}u$$

Draft of the inverse problem

$$\begin{cases} (\partial_t^2 + (-\Delta)^s)u + f(x,u) = 0, & \text{in } \Omega_T \\ u = \varphi & \text{in } (\Omega_e)_T \\ u(0) = \partial_t u(0) = 0 & \text{in } \Omega \end{cases}$$

Exterior data fixes u and the DN-map $(-\Delta)^s u$ in W_2, so UCP determines u in Ω_T, which finally determines

$$f(x,u) \quad \text{a.e. } (x,t) \in \Omega_T.$$

It only remains to take suitable solutions u to conclude that

$$f(x,\tau)$$

is determined for almost all $x \in \Omega$ and $\tau \in \mathbb{R}$.

Recovering the leading nonlinear term

Let u_ε solve the nonlinear equation with small exterior data

$$\varphi_\varepsilon = \varepsilon\varphi, \quad \mathrm{supp}(\varphi) \subset W_1.$$

From the energy estimates it follows that

$$u_\varepsilon = \varepsilon v + R_\varepsilon,$$

where v solves the linearized equation

$$(\partial_t^2 + (-\Delta)^s)v = 0$$

with $v = \varphi$ in W_1 and

$$R_\varepsilon = O(\varepsilon^{r_1+1} + \varepsilon^{r_\infty+1}) \quad \text{in } L^\infty(0, T; \widetilde{H}^s(\Omega))$$

Recovering the leading nonlinear term

The corresponding solutions satisfy

$$\frac{u_\varepsilon}{\varepsilon} = \frac{\varepsilon v + R_\varepsilon}{\varepsilon} \xrightarrow{\varepsilon \to 0} v \quad \text{in } L^q(0, T; L^{p'}(\Omega))$$

with big enough q and p' the critical Sobolev exponent of H^s.

Using the homogeneity of the leading term:

$$\frac{f(x, u_\varepsilon) - f_1(x, u_\varepsilon)}{\varepsilon^{r_1+1}} = \frac{f(x, u_\varepsilon)}{\varepsilon^{r_1+1}} - f_1\left(x, \frac{u_\varepsilon}{\varepsilon}\right) = O(\varepsilon^{r_2 - r_1})$$

so higher order terms disappear in the limit $\varepsilon \to 0$ and

$$\frac{f(x, u_\varepsilon)}{\varepsilon^{r_1+1}} \xrightarrow{\varepsilon \to 0} f_1(x, v) \quad \text{in } L^{\frac{q}{r_\infty+1}}(0, T; L^{\frac{p'}{r_\infty+1}})$$

Runge approximation and identification

Runge approximation: pick $\Phi(x,t) = \eta(t)\Psi(x)$ s.t. $\eta(t) = 1$ near $t = t_0$ and $\Psi(x) = 1$ near $x = x_0$ (smooth, compact support), and choose solutions v_j such that

$$v_j \to \Phi(x,t) \quad \text{in } L^2(0,T;\widetilde{H}^s(\Omega)).$$

Passing to the limit using Sobolev embeddings and mapping properties of Nemytskii operators

$$f_1(x, v_j) \longrightarrow f_1(x, 1)$$

Homogeneity then implies recovery for all amplitudes:

$$\tau^{r_1+1} f_1(x,1) = f_1(x,\tau)$$

Hence the leading nonlinear term is uniquely determined.

Subtracting the recovered term and repeating the argument inductively recovers the full polyhomogeneous expansion.


Thank you!

References

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- Lin Y-H, Tyni T, and Zimmermann P, Well-posedness and inverse problems for semilinear nonlocal wave equations, Nonlinear analysis, 247, 113601, 2024.