



Inverse scattering problems for nonlinear equations on Lorentzian manifolds

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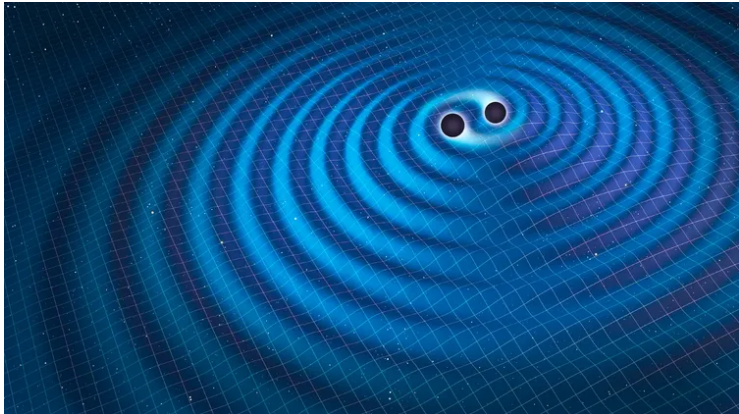
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Motivation for the scattering problem

Can one recover the spacetime structure from scattering data?



(Image credit: Mark Garlick/Science Photo Library/Getty Images)

Motivation for the scattering problem

Can one recover the spacetime structure from scattering data?

A couple of examples

- ▶ Sá Barreto (Duke 2005): if space part of spacetime is asymptotically hyperbolic and setting is time-independent (static), then Carleman estimates and boundary control yield unique determination of metric: $\Phi^* g_1 = g_2$
- ▶ Sá Barreto, Wang, Uhlmann (Pure Appl. Anal. 2022): nonlinear scattering. Potential recovery for $(\partial_t^2 - \Delta)u + f(u)$, $f(u) \sim u^5$, via Melrose-type compactification (stereographic projection to compactify space)

Motivation for the scattering problem

Can one recover the spacetime structure from **nonlinear** scattering data?

Yes:

Theorem (Alexakis, Isozaki, Lassas, T.)

Let (N, g) be a $3 + 1$ dimensional Lorentzian manifold that has an asymptotically Euclidean infinity. The scattering data related to

$$\square_g u + au^4 = 0, \quad a \neq 0,$$

uniquely determines the conformal class of the metric g , the topology of N , and the potential a , up to a gauge.

Brief history of nonlinear inverse problems

- ▶ Kurylev, Lassas, Uhlmann (2014-2018): Using nonlinearity and higher order linearization to solve inverse problems
- ▶ Hyperbolic inverse problems (Feizmohammadi, Kian, Lassas, Liimatainen, Oksanen, Stefanov, Uhlmann, Y Wang, ...)
- ▶ Elliptic problems (Cârstea, Johansson, Krupchyk, Y-H Lin, Nurminen, Rüland, Salo, ...)
- ▶ Fractional problems (Covi, Lin, Salo, Koch, Railo, Rüland, Zimmermann, ...)
- ▶ and many others!

Also: Hintz, Sá Barreto, Uhlmann, Zhang: *Inverse Nonlinear scattering by metric* (arxiv 2024, on the same day as the paper I'm discussing)

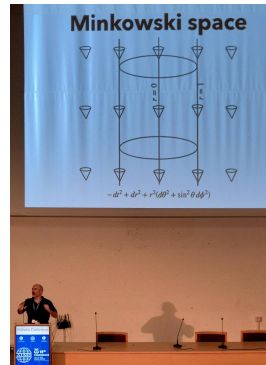
Penrose conformal compactification

Let $(\mathbb{R}^{1+3}, \eta)$ be the Minkowski space with its standard metric η in polar coordinates (t, r, θ, ϕ) :

$$\eta = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2(\theta)d\varphi^2)$$

We make a conformal change to $\tilde{\eta} := \Omega^2 \eta$ with

$$\Omega = 4 \frac{1}{1 + (t + r)^2} \frac{1}{1 + (t - r)^2}$$



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Penrose conformal compactification

- ▶ Let $\tilde{\eta} = \Omega^2 \eta$, where

$$\Omega = 4 \frac{1}{1 + (t + r)^2} \frac{1}{1 + (t - r)^2}$$

- ▶ $\Phi : \mathbb{R}^{1+3} \rightarrow \mathbb{R} \times \mathbb{S}^3$, defined by

$$\Phi(t, r, \theta, \varphi) = (T, R, \theta, \varphi),$$

where

$$\begin{aligned} T &= \arctan(t + r) + \arctan(t - r), \\ R &= \arctan(t + r) - \arctan(t - r), \\ -\pi &< T + R < \pi, \quad -\pi < T - R < \pi, \quad R \geq 0. \end{aligned}$$

On $\mathbb{S}^3$ we have the standard spherical coordinates (R, θ, φ) , and the metric on the cylinder $\mathbb{R} \times \mathbb{S}^3$ is of the form

$$g_{\mathbb{R} \times \mathbb{S}^3} = -dT^2 + dR^2 + \sin^2(R) (d\theta^2 + \sin^2(\theta) d\varphi^2).$$

Penrose conformal compactification

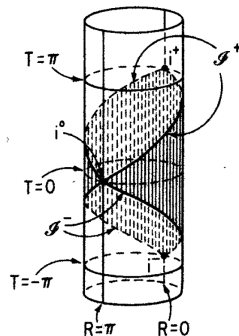
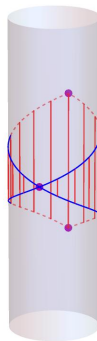
Thus

$$\Phi : \mathbb{R}^{1+3} \rightarrow \mathbb{R} \times \mathbb{S}^3$$

is conformal diffeomorphism. We call

$$\hat{N} = \Phi(\mathbb{R}^{1+3}) \subset \mathbb{R} \times \mathbb{S}^3$$

the **Penrose diagram** of $\mathbb{R}^{1+3}$ and Φ the Penrose map.



Right picture: R. Wald *General relativity*, 1984

Penrose conformal compactification

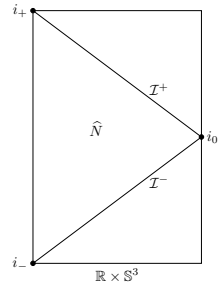
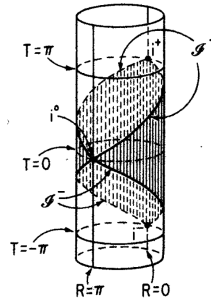
Thus

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is conformal (iso-metric) diffeomorphism. We call

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the Penrose diagram of $\mathbb{R}^{1+3}$.



Notation for the wave equation

Let $(\mathbb{R}^{1+3}, g)$ be a globally hyperbolic Lorentzian manifold and consider the nonlinear wave equation

$$\square_g u(t, y) + a(t, y)u(t, y)^\kappa = 0, \quad (t, y) \in \mathbb{R}^{1+3}.$$

where $\kappa \geq 4$ is an integer and $a(t, y)$ a smooth rapidly decaying function.

Here $\square_g$ is the D'Alembertian wave operator

$$\square_g u = - \sum_{a,b=0}^n \frac{1}{\sqrt{|\det(g)|}} \frac{\partial}{\partial x^a} \left(\sqrt{|\det(g)|} g^{ab} \frac{\partial u}{\partial x^b} \right)$$

Towards a scattering problem

- ▶ Let η be the standard Minkowski metric on $\mathbb{R}^{3+1}$.
- ▶ Let g be a globally hyperbolic Lorentzian metric on $\mathbb{R}^{3+1}$, such that $g_{ij}(x) - \eta_{ij}$ is a Schwartz rapidly decaying function and
- ▶ Let $\tilde{g} = \Omega^2 g$ be a conformal metric to g and let $\widehat{g} = \Phi_* \tilde{g}$ be the pushforward metric on the Penrose diagram.

Then u satisfies the nonlinear wave equation

$$\square_g u + a u^\kappa = 0$$

in $\mathbb{R}^{3+1}$ iff $\tilde{u} = (\Omega^{-1} u) \circ \Phi^{-1}$ satisfies

$$(\square_{\widehat{g}} + B)\tilde{u} + A\tilde{u}^\kappa = 0$$

in $\widehat{N}$, where

$$A := (\Phi^{-1})^*(a\Omega), \quad B := \frac{1}{6}(\Phi^{-1})^*(R_{\Omega^2 g} - \Omega^{-2} R_g).$$

A geometric scattering problem

We say that a function $u \in H_{\text{loc}}^m(\mathbb{R} \times \mathbb{R}^3)$ is a solution of the scattering problem on $(\mathbb{R} \times \mathbb{R}^3, g)$, with the past radiation field h_- ,

$$\begin{cases} \square_g u(x) + a(x) \cdot u(x)^k = 0, & \text{in } \mathbb{R} \times \mathbb{R}^3, \\ u(x) \sim h_-(x) & \text{as } x \text{ goes to } \mathcal{I}^- \end{cases}$$

if the function $\tilde{u} = (\Omega^{-1}u) \circ \Phi^{-1}$ satisfies $\tilde{u} \in H^m(\hat{N})$ and it is a solution of the Goursat-Cauchy boundary value problem

$$\begin{cases} \square_{\hat{g}} \tilde{u}(x) + B(x)\tilde{u}(x) + A(x) \cdot \tilde{u}(x)^k = 0, & \text{in } \hat{N}, \\ \tilde{u}|_{\mathcal{I}^-} = \tilde{h}_-, \end{cases}$$

Scattering problem has a solution

Lemma

Let g be a globally hyperbolic Lorentzian metric on $\mathbb{R}^{3+1}$, where $g - \eta$ is in the Schwartz class. Let $(\hat{N}, \Omega^2 g)$ be the Penrose diagram. Let $-\pi < T_- < 0 < T_+ < \pi$. There is $\varepsilon > 0$ such that the following holds: Let $h \in H^m(\mathcal{I}^-)$ be such that $\text{supp}(h) \subset \{x \in \mathcal{I}^- \mid t(x) > T_-\}$ and $\|h\|_{H^m(\mathcal{I}^-)} < \varepsilon$ (m large enough). Then the non-linear scattering problem

$$\begin{cases} (\square_g + B)u + Au^k = 0, & \text{in } \{x \in \hat{N} : t(x) < T_+\}, \\ u|_{\mathcal{I}^-} = h, \\ u = 0 \text{ in } \{x \in \hat{N} : t(x) < T_-\} \end{cases} \quad (1)$$

has a unique solution depending continuously on h

Defining the scattering operator

We define the (geometric) scattering operator on $\hat{N}$ by

$$\begin{aligned} S : C_c^\infty(\mathcal{I}^-) \supset U &\rightarrow C^\infty(\mathcal{I}^+), \\ S(u|_{\mathcal{I}^-}) &= u|_{\mathcal{I}^+}, \quad u = h \in U, \end{aligned}$$

for a neighbourhood U of the zero function in $C_c^\infty(\mathcal{I})$. Here u solves the (non-linear) scattering problem

$$\begin{cases} (\square_g + B)u + Au^\kappa = 0, & \text{in } \{x \in \hat{N} : t(x) < T_+\}, \\ u|_{\mathcal{I}^-} = h, \\ u = 0 \text{ in } \{x \in \hat{N} : t(x) < T_-\} \end{cases} \quad (2)$$

Inverse scattering problem

Theorem (Alexakis, Isozaki, Lassas, T.)

If $\text{supp}(a) = \mathbb{R}^{1+3}$, the non-linear scattering operator S , defined in a neighborhood of the zero function in $C_0^\infty(\mathcal{I}^-)$, determines the conformal class of g .

If $a \equiv 1$, the conformal factor can also be determined.

A non-physical extension

For simplicity, let's work on $(\mathbb{R}^3 \times \mathbb{R}, g)$.

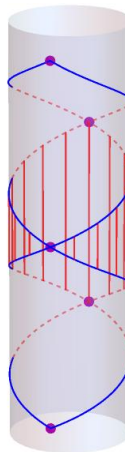
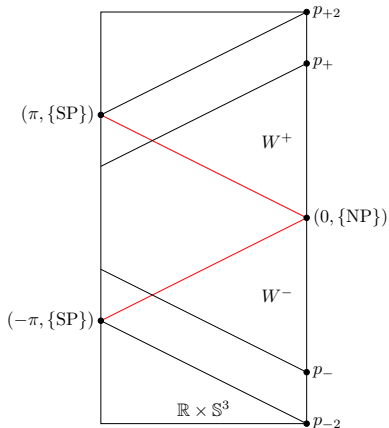
Assume that

- ▶ the metric g is globally hyperbolic Lorentzian metric and $g_{ij} - \eta_{ij}$ belong to the Schwartz class
- ▶ $\hat{g} := \Phi_* \tilde{g}$ is the push-forward metric of $\tilde{g} := \Omega^2 g$ on the Penrose diagram

Then $\hat{g}$ can be smoothly extended to $\mathbb{R} \times \mathbb{S}^3$ by defining $g_e = \hat{g}$ in $\hat{N}$ and $g_e = \eta$, for $x \in \mathbb{R} \times \mathbb{S}^3 \setminus \hat{N}$

Measurements beyond infinity

We can do an artificial extension of the Penrose diagram $\hat{N}$ by gluing it into the cylinder $\mathbb{R} \times \mathbb{S}^3$:



Scattering operator determines a source-to-solution map

Solve a linear wave equation

$$\begin{cases} (\square_{g_e} + B)u = f, \\ \text{supp}(u) \subset J^+(\text{supp}(f)). \end{cases} \quad \text{in } N_{\text{ext}},$$

up to $\mathcal{I}^-$.

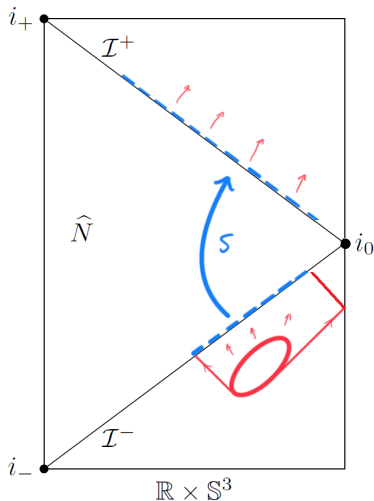
Scattering operator determines

$$h^+ := u|_{\mathcal{I}^+} = S(u|_{\mathcal{I}^-}).$$

Finally, solve

$$\begin{cases} (\square_{g_e} + B)u = 0, & \text{in } N_{\text{ext}}, \\ u|_{\mathcal{I}^+} = h^+, \end{cases}$$

to determine u in the non-physical future.



Sketch of the Inverse Problem: Higher order linearization

A k -fold linearization of the nonlinear equation

$$(\square_g + B)u + Au^k = \sum_{j=1}^k \varepsilon_j f_j \quad (3)$$

with respect to ε_j yields

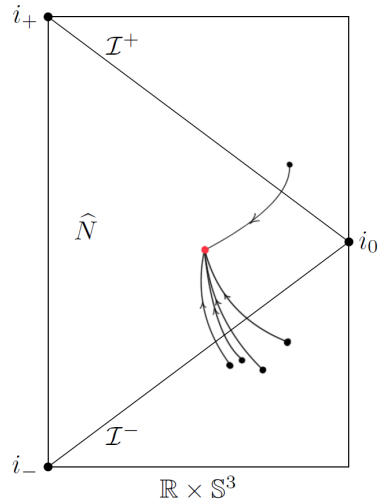
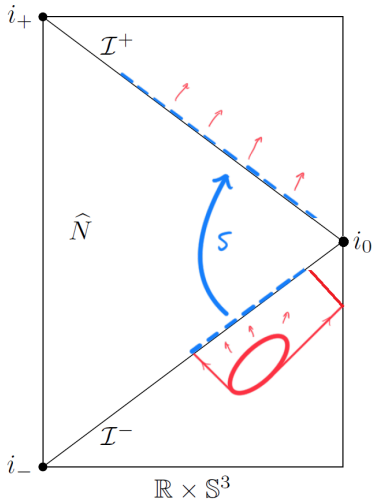
$$(\square_g + B)w + Av_1 v_2 \cdots v_k = 0 \quad (4)$$

where

$$(\square_g + B)v_j = f_j, \quad \text{in } \hat{N}$$

The products $v_1 v_2 \cdots v_k$ can be used to produce point sources: if v_j , $j = 1, 2, 3, 4$ propagate on transversal Lagrangians, before conjugate points they meet at most at one point, where their interaction produces a point source.

Nonlinear interactions

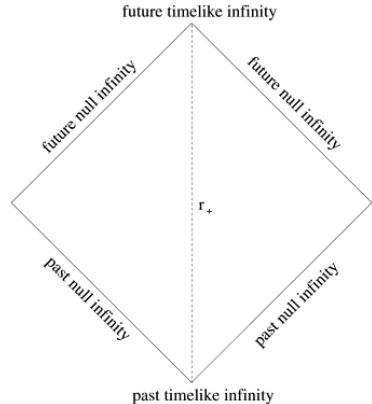
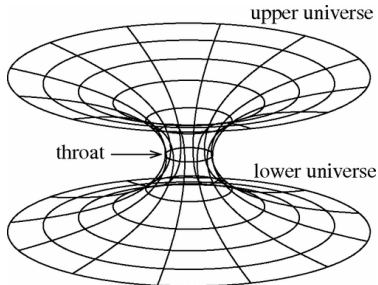


Four-wave interaction causes singularity, which we detect in the future

Rough sketch of the inverse scattering problem

- ▶ Detect the earliest singularities arriving from nonlinear interactions
- ▶ Observations of arriving light determine parts of lightcones. . .
- ▶ which determine the full lightcones
- ▶ Lightcones determine the metric up to a conformal factor
- ▶ The principal symbol of arriving nonlinear waves carries the coefficient a .

More general manifolds are allowed



T Muller, Exact geometric optics in a Morris-Thorne wormhole spacetime Physical review D, 20(2008), 044043

R. Garattini, F. Lobo Self-sustained traversable wormholes in noncommutative geometry Physics Letters B 671(2009):146-152

Thank you!



Reference

- ▶ Alexakis S, Isozaki H, Lassas M, and Tyni T, Inverse scattering problems for non-linear wave equations on Lorentzian manifolds, arXiv:2411.09354, 2024.